Estimation of initial conditions from a scalar time series

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We introduce a method to estimate the initial conditions of a mutivariable dynamical system from a scalar signal. The method is based on a modified multidimensional Newton-Raphson method which includes the time evolution of the system. The method can estimate initial conditions of periodic and chaotic systems and the required length of scalar signal is very small. Also, the method works even when the conditional Lyapunov exponent is positive. An important application of our method is that synchronization of two chaotic signals using a scalar signal becomes trivial and instantaneous.

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A trajectory that a given dynamical system traverses in its state space depends on the particular set of initial conditions with which it starts. In particular, the state of a chaotic system at a latter time is exponentially sensitive to changes in its initial state [1]. This defining feature of a chaotic system leads to a complex behaviour in state space that appears random yet is deterministic, which means that an initial state uniquely fixes the future course of its evolution. Though their are several invariant measures of a chaotic system which are not sensitive to the intial conditions, the exact trajectory crucially depends on the intial state and hence is difficult to reproduce due to sensitivity to initial conditions.

In light of these facts, it is interesting and important to ask whether the complete set of initial conditions of a given multivariable dynamical system can be estimated from a given scalar time series for a single state space variable. We show that this question can be answered in the affirmative and present a novel and simple method to estimate the initial conditions. Our method is based on a modified multidimensional Newton-Raphson method [1,2] where we include the time evolution of the system. The length of the time series required for the calculations is typically very small.

Our results raise some interesting issues regarding the information content of a time series. In standard embedding techniques, a vector space is constructed from successive iterates of a single variable and a trajectory is reconstructed in this space [3]. While embedding, it is crucial to choose an appropriate time delay so that the successive iterates are well resolved and contain qualitatively different information. In our method we use a very small time series and the total duration is typically much less than the standard delay time in embedding techniques. It is interesting that we can recover the initial conditions and hence the trajectory from such a small duration of time series.

An important application of our method is in the problem of synchronization of two chaotic systems [4]. This problem itself has attracted wide attention in recent times due to its potential application to secured communication [5–7] and parameter estimation [8,9]. Our estimation of the initial conditions makes the problem of synchronization almost trivial. We also find that our method works for most of the cases where other methods fail [4,10].

Let us consider an autonomous dynamical system given by,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ is a d-dimensional state vector whose evolution is governed by the function $\mathbf{F} = (F_1, F_2, \ldots, F_d)$. Given an initial state vector $\mathbf{x}(0)$ at time $t = 0$, the time evolution $\mathbf{x}(t)$ is uniquely determined by Eq. (1). Now let us assume that only one component of the state vector is known to us and we take it to be $x_1(t)$ without loss of generality. The problem that we address is to obtain the initial state vector $\mathbf{x}(0)$ from the knowledge of the scalar signal $x_1(t)$.

Let $\mathbf{y}(0)$ denote a random initial state vector and $\mathbf{y}(t)$ its time evolution obtained from Eq. (1). Let $\mathbf{w}(t)$ denote the difference

$$\mathbf{w}(t) = \mathbf{y}(t) - \mathbf{x}(t). \tag{2}$$

We look for the solution of the equation

$$\mathbf{w}(t) = 0. \tag{3}$$

Noting that the initial state vectors $\mathbf{y}(0)$ and $\mathbf{x}(0)$ uniquely determine the difference $\mathbf{w}(t)$, one of the solutions of Eq. (3) is $\mathbf{y}(\mathbf{0}) - \mathbf{x}(0) = 0$ and this is the solution that we are searching for.

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We now introduce the notation $\mathbf{w}^n = \mathbf{w}^n(\mathbf{y}^0, \mathbf{x}^0) = \mathbf{w}(n\Delta t)$, where Δt is a small time interval. Similarly, $\mathbf{y}^n = \mathbf{y}(n\Delta t)$ and $\mathbf{x}^n = \mathbf{x}(n\Delta t)$. With this notation condition (3) can be written as $\mathbf{w}^n = 0$.

Our approach to the solution of Eq. (3) is a modified Newton-Raphson method [1] which includes the time evolution of the system.

Let us first consider $\mathbf{w}^1$. We have

$$\begin{aligned} 0 &= \mathbf{w}^1(\mathbf{x}^0, \mathbf{x}^0), \\ &= \mathbf{w}^1(\mathbf{y}^0 + \delta\mathbf{y}^0, \mathbf{x}^0), \\ &= \mathbf{w}^1(\mathbf{y}^0, \mathbf{x}^0) + (\delta\mathbf{y}^0 \cdot \boldsymbol{\nabla}_{\mathbf{y}^0})\mathbf{w}^1(\mathbf{y}^0, \mathbf{x}^0) + \mathcal{O}((\delta\mathbf{y}^0)^2), \end{aligned} \quad (4)$$

where $\delta\mathbf{y}^0 = \mathbf{x}^0 - \mathbf{y}^0 = -\mathbf{w}^0$ and the last step is a Taylor series expansion in $\delta\mathbf{y}^0$. For small Δt, we can write

$$\mathbf{w}^1(\mathbf{y}^0, \mathbf{x}^0) = \mathbf{w}^0 + \Delta t\,[\mathbf{F}(\mathbf{y}^0) - \mathbf{F}(\mathbf{x}^0)] + \mathcal{O}((\Delta t)^2). \quad (5)$$

Substituting Eq. (5) in Eq. (4) and neglecting higher order terms, we get,

$$\mathbf{w}^1(\mathbf{y}^0, \mathbf{x}^0) = \mathbf{w}^0 - \Delta t(\mathbf{w}^0 \cdot \boldsymbol{\nabla}_{\mathbf{y}^0})\mathbf{F}(\mathbf{y}^0). \quad (6)$$

It is convenient to write the above equation in a matrix form as

$$\begin{aligned} W^1 &= (I + \Delta t\, J^0)W^0, \\ &= A^0 W^0, \end{aligned} \quad (7)$$

where W^n is the column matrix corresponding to the vector $\mathbf{w}^n$, I is the identity matrix, $A^n = I + \Delta t\, J^n$, and the elements of the Jacobian matrix J^n are given by

$$J_{ij}^n = \frac{\partial F_i(\mathbf{y}^n)}{\partial y_j^n}. \quad (8)$$

Next we consider $\mathbf{w}^2$ or W^2. Proceeding as above, we get (see Eq. (7)),

$$\begin{aligned} W^2 &= (I + \Delta t\, J^1)W^1 \\ &= (I + \Delta t\, J^1)(I + \Delta t\, J^0)W^0 \\ &= A^1 A^0 W^0. \end{aligned} \quad (9)$$

Similarly, the equation for W^n is

$$\begin{aligned} W^n &= (I + \Delta t\, J^{n-1})(I + \Delta t\, J^{n-2}) \cdots (I + \Delta t\, J^0)W^0 \\ &= A^{n-1}A^{n-2} \cdots A^0 W^0. \end{aligned} \quad (10)$$

We now concentrate on the first component of the signal whose time series is assumed to be known. For a d-dimensional system we need $d-1$ equations to determine the initial state vector $\mathbf{x}^0$. Eqs. (7), (9) and (10) give us the required relations.

$$\begin{aligned} W_1^1 &= \sum_{i=1}^{d} A_{1i}^0 W_i^0, \\ W_1^2 &= \sum_{i,j=1}^{d} A_{1i}^1 A_{ij}^0 W_j^0, \\ &\vdots \\ W_1^{d-1} &= \sum_{i,\ldots,l,m=1}^{d} A_{1i}^{d-2} \cdots A_{lm}^0 W_m^0, \end{aligned} \quad (11)$$

These are $d-1$ simultaneous equations for W^0.

The numerical procedure is as follows. We set the intial state of system (1) to a random initial guess vector $(\mathbf{y}^0)_{old}$ with $(y_1^0)_{old} = x_1^0$ and evolve it using Eq. (1). Using this vector $\mathbf{y}(t)$ we write down $d-1$ simultaneous equations (Eqs. (11)) which can be solved for $d-1$ unknown components of $\mathbf{w}^0 = -\delta\mathbf{y}^0$. Also, $\delta y_1^0 = 0$. Thus the initial guess vector can be improved by

$$(\mathbf{y}^0)_{new} = (\mathbf{y}^0)_{old} + \delta\mathbf{y}^0. \quad (12)$$

This sets up an iterative scheme giving us better and better estimates of the initial vector which converge to $\mathbf{x}^0$.

We note that as in Newton-Raphson method, the choice of the initial guess vector can be very important [2]. In some cases, the iterative procedure of Eq. (12) may not converge or converge to a wrong root. In such cases, a different choice of initial guess vector can be useful.

We further note the similarity of our method with the so called method of variational equations in analytical dynamics [11]. The method of variational equations can be appied to a known Hamiltonian system to determine an unknown neighbouring trajectory to an already known one. There, the method requires a complete particular solution of a known set of Hamiltonian equations of motion. In contrast, we have used our method for dissipative chaotic systems. In such systems an analytical solution of the equations of motion cannot be known. Further our method requires only one component of a complete trajectory to be sampled. This has important consequences in the problem of synchronization using a scalar signal.

We now demonstrate our method of estimating the initial state. As our first example we discuss the Rössler system given by [12],

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + a x_2, \\ \dot{x}_3 &= b + x_3(x_1 - c). \end{aligned} \quad (13)$$

First, we consider a case when the time series for x_1 is given and we want to estimate (x_2^0, x_3^0). We chose the parameters (a, b, c) such that the system is in the chaotic regime and the initial state $\mathbf{x}^0$ is on the chaotic attractor. We start with an arbitrary initial state $\mathbf{y}^0 = (y_1^0, y_2^0, y_3^0)$ with $y_1^0 = x_1^0$. From Eqs. (11) we get a pair of simultaneous equations as,

$$
\begin{aligned}
w_1^1 &= y_1^1 - x_1^1 = \Delta t \delta y_2^0 + \Delta t \delta y_3^0 \\
w_1^2 &= y_1^2 - x_1^2 = \left(2\Delta t + a(\Delta t)^2\right) \delta y_2^0 \\
&\quad + \left(2\Delta t + (y_1^0 - c)(\Delta t)^2\right) \delta y_3^0 \qquad (14)
\end{aligned}
$$

which can be solved for $(\delta y_2^0, \delta y_3^0)$. With $\delta y_1^0 = 0$ we use these in an iterative manner (Eq. (12)) to obtain the correct intial conditions.

Table 1 shows successively corrected (y_2^0, y_3^0) obtained using the iterative process as discussed. These are the successive estimates for (x_2^0, x_3^0). Let $e_i = |y_i^0 - x_i^0|$ denote the absolute error in the estimation of x_i^0. In Fig. 1(a) we plot a graph corresponding to Table 1 showing errors e_2 and e_3 (on logarithmic scale) plotted against the number of iterations of our method (Eq. (12)). From Table 1 and Fig. 1(a) we see that the successive estimates converge to the correct values of (x_2^0, x_3^0). Using only two data points in the given time series $x_1(t)$, we can thus readily estimate the full initial state $\mathbf{x}^0$. We also note that the rate of convergence is very good. In about 8 to 10 iterates we obtain the initial values (x_2^0, x_3^0) to within computer accuracy. If we write the deviations of the succesive iterates from the correct values in the form

$$
(e_i)_n = \left|\left(y_i^0\right)_n - x_i^0\right| \sim e^{-\alpha n}, \qquad (15)
$$

where n is the number of iterations, then the value of the parameter α is found to be 2.01 for e_2 and 2.02 for e_3. This is consistant with the fact that Newton-Raphson method has a quadratic convergence [1,2].

We note that the largest Lyapunov exponent for the subsystem (y_2^0, y_3^0) (conditional or subsystem Lyapunov exponent) is positive [4]. The success of our method does not depend on whether this Lyapunov exponent is positive or negative. This is important for synchronization of chaotic signals as will be discussed afterwards.

We next present cases where time series for the variables x_2 and x_3 of the Rössler system are given. The procedure is similar to the case of time series for x_1 as discussed above. Fig. 1 (b) shows the errors e_1 and e_3, when time series for x_2 is given, plotted against the number of iterations. The parameter α (Eq. (15)) is 1.97 for e_1 and 1.95 for e_3. This again indicates a quadratic convergence. Similarly, Fig. 1 (c) shows the quantities e_1 and e_2 when time series for x_3 is given, as a function of the number of iterations. The parameter α (Eq. (15)) is 1.28 for e_1 and 1.30 for e_2 which shows a convergence slower than quadratic. We note that the largest subsystem Lyapunov exponent is negative when time series for x_2 is given and is positive when time series for x_3 is given [4].

As our next example we consider the Chua's circuit which in its dimensionless form is given by [13],

$$
\begin{aligned}
\dot{x}_1 &= \alpha(x_2 - x_1 - f(x_1)), \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta x_2,
\end{aligned}
$$

$$
f(x_1) = b x_1 + \frac{1}{2}(a - b)\left[|x_1 - 1| - |x_1 + 1|\right]. \qquad (16)
$$

We chose the parameters a, b, α and β such that the attractor is a limit cycle. The initial state $\mathbf{x}^0$ is chosen in the basin of attraction of this limit cycle. Fig. 2 shows the errors e_2 and e_3 when a time series for the variable x_1 is given as a function of the number of iterations of our method. The parameter α (Eq. (15)) in this case takes values 1.29 for e_2 and 1.25 for e_3 showing a slower than quadratic convergence.

We have also applied our method for the cases when time series for the variables x_2 and x_3 are given and also for cases when parameters are such that the attractor is chaotic. In all the cases we are able to estimate the full initial state vector.

We have successfully applied our method to estimate the initial state vector using a given scalar time series for many other dynamical systems as well. These include Lorenz system [14] in its periodic, chaotic or intermittent regimes, the disk dynamo system modelling a periodic reversal of earth's magnetic field [15,16], a 3-d plasma system formed by a three wave resonant coupling equations [17] and a four dimensional phase converter circuit [18].

Now we will discuss an important application of our estimation method in the problem of synchronization of two identical chaotic systems coupled unidirectionally by a scalar signal. Let us suppose that Eq. (1) describes a chaotic system and let us consider a replica of it given by, $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y})$. Without losing generality let's further assume that a scalar output signal $x_1(t)$ is given. The aim is to synchronize vector $\mathbf{y}(t)$ with $\mathbf{x}(t)$ using this scalar signal. Our method of estimating initial conditions makes this procedure trivial. Using the scalar signal we estimate the initial vector $\mathbf{x}^0$ and set $\mathbf{y}^0 = \mathbf{x}^0$. This clearly leads to an instantaneous synchronization of the two trajectories. As we have demonstrated in the case of Rössler system, our method works even when the largest conditional Lyapunov exponent is positive which is the case when other methods of synchronization are known to fail [4,10].

To summarise, we have introduced a novel yet simple method to estimate initial conditions of a multivariable dynamical system from a given scalar signal. Our method is based on a multidimensional Newton-Raphson method where we include the time evolution of the system. The method gives a reasonably fast convergence to the correct initial state. The required length of the time series is very small. The method works even when the largest conditional Lyapunov exponent is positive. An important consequence of the method is that the problem of synchronization of identical chaotic systems using scalar signal becomes trivial since evolution of two such systems can then be started from identical initial states.

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TABLE I. The table presents successive estimates of the unknown components of the initial state vector, i.e. (y_2^0, y_3^0), in a Rössler system for which a time series $x_1(t)$ is given. n is the number of iterations of our method (Eq. (12)). A clear convergence is seen towards the actual values x_2^0, x_3^0, written at the bottom of the table. The parameters are $(a, b, c) = (0.2, 0.2, 9.0)$. The time step Δt is 0.01 and we use fourth order Runge-Kutta method for time evolution of Eq. (13). The corresponding plots are shown in Fig. 1(a).

n	y_2^0	y_3^0
0	-9.168384973984731	7.4923642142341230
1	-5.166404733407783	0.4244936662730687
2	-5.178027081551812	0.4761293644577211
3	-5.178176377100173	0.4759664825328775
4	-5.178173198394843	0.4759644699505680
5	-5.178173232249095	0.4759645133108157
6	-5.178173232007102	0.4759645128302578
7	-5.178173232003735	0.4759645128297332
8	-5.178173232003735	0.4759645128297332
	$x_2^0 = -5.178173232007801$	$x_3^0 = 0.4759645128337372$

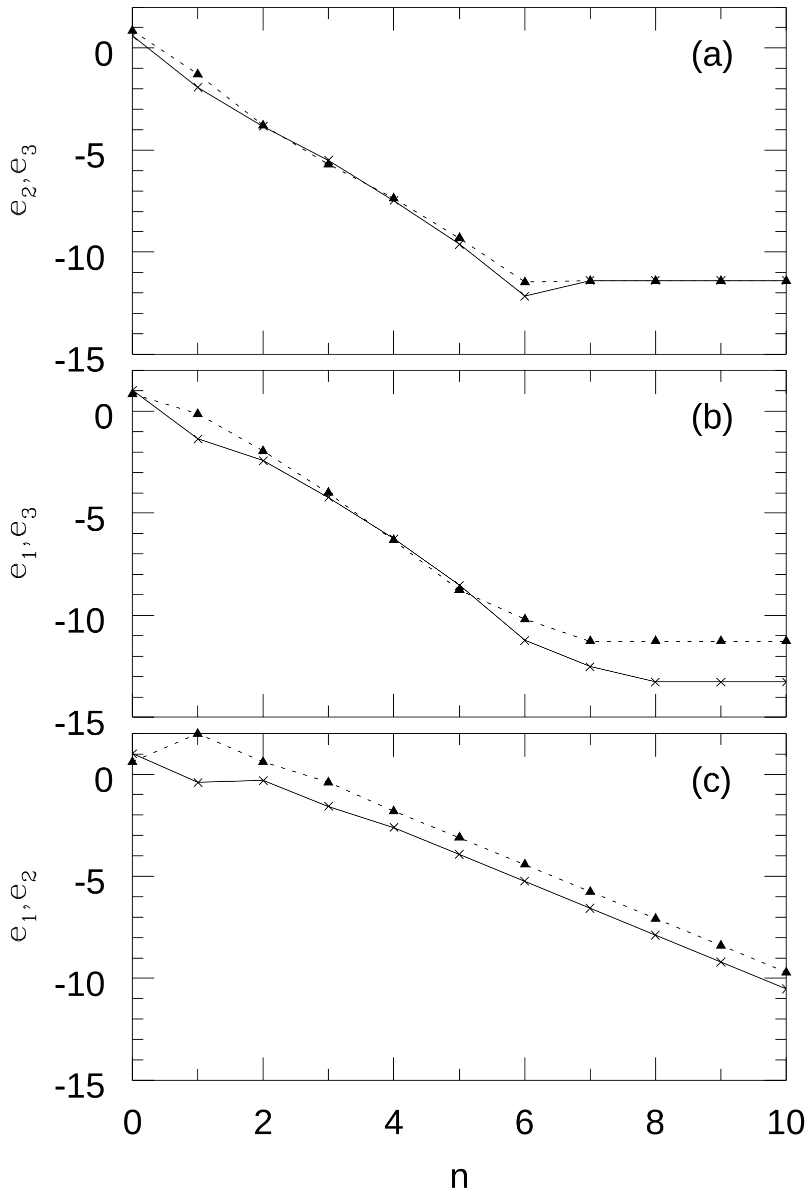

Fig.1 (AM and REA)

FIG. 1. Plot (a) shows the errors e_2 and e_3 on a logarithmic scale, as a function of n, the number of iterations of our method for Rössler system when time series for variable x_1 is given. The crosses show the values for e_2 and solid triangles those for e_3. Errors are seen to approach zero as n increases. The values are taken from Table 1. Similarly plot (b) shows the errors e_1 (crosses), and e_3 (solid triangles) as a function of n when the time series for x_2 is given. Plot (c) shows the errors $e_|$ (crosses), and e_2 (solid triangles) as a function of n when the time series for x_3 is given. For all these cases, the parameters are such that the attractor is chaotic and the initial state is on the attractor.

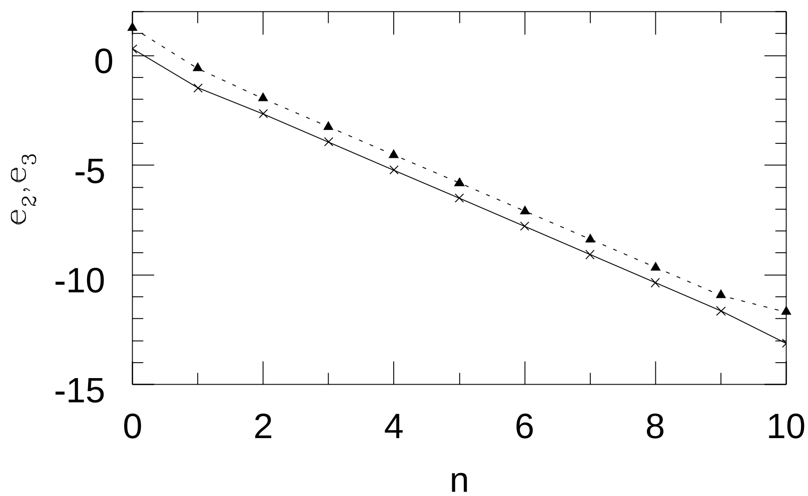

Fig.2 (AM and REA)

FIG. 2. The figure shows errors e_2 and e_3 on a logarithmic scale, as a function of n, the number of iterations of our method for Chua's circuit when time series for the variable x_1 is given. The crosses show the values for e_2 and solid triangles those for e_3. Errors are seen to approach zero as n increases. The parameters are$(4, 3, 2, 0.1758)$ and are such that the attractor is a limit cycle and the initial state is in the basin of the attractor.